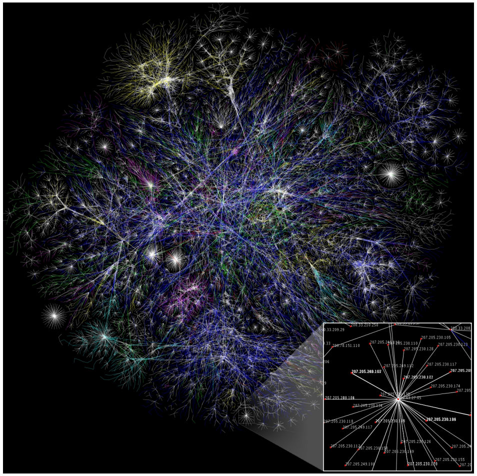


Scale-free percolation

Remco van der Hofstad

Simons Conference on Random Graph Processes,
May 9–12, 2016, UT Austin

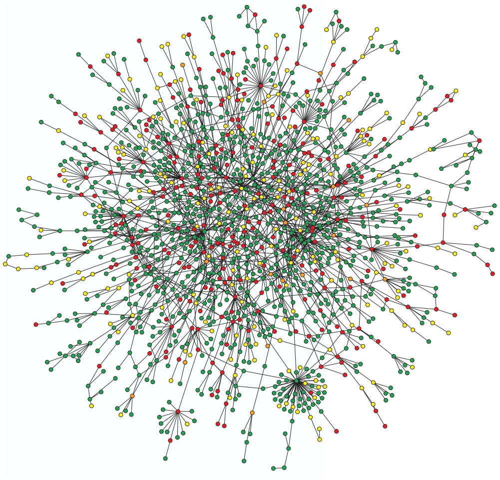
Technische Universiteit Eindhoven University of Technology

EURANDOM

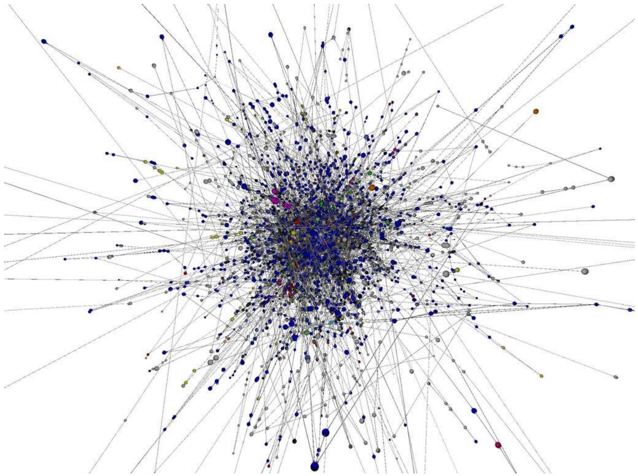
NETWORKS

Joint work with:
▷ Mia Deijfen (Stockholm)
▷ Gerard Hooghiemstra (TU Delft)

Complex networks



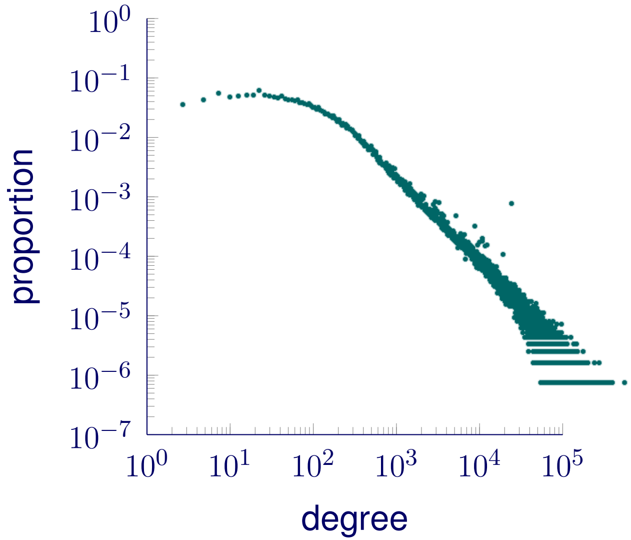
Yeast protein interaction network

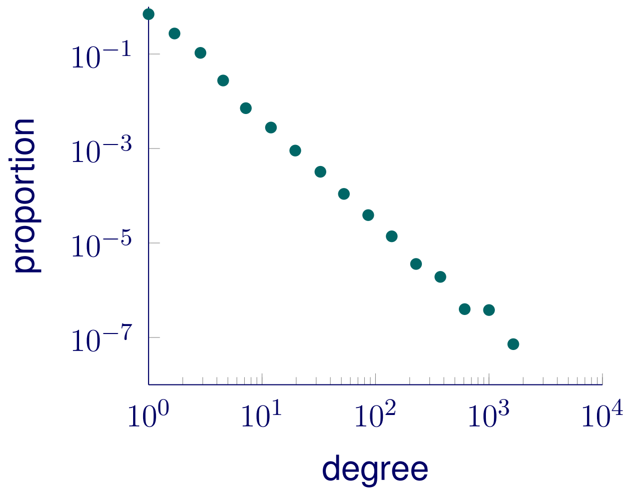


Internet topology in 2001

Attention focussing on unexpected commonality.

Scale-free paradigm



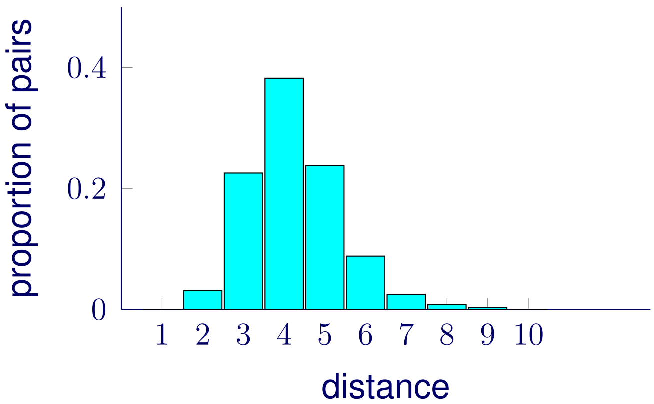


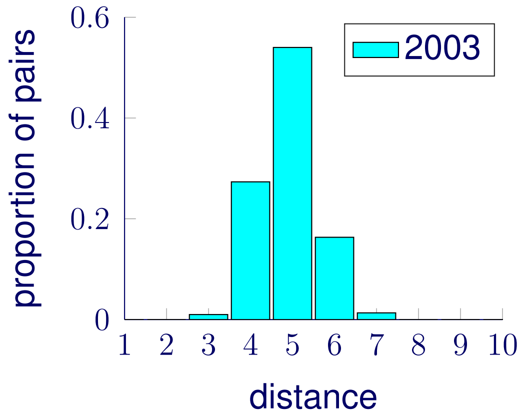
Loglog plot degree sequences Internet Movie Database and Internet

▷ Straight line: proportion p_k of vertices with degree k satisfies

$$p_k = ck^{-\tau}.$$

Small-world paradigm





Distances in SCC WWW and IMDb in 2003.

Random graphs for complex networks

$\triangleright$ Inhomogeneous random graph:
Vertex set $[n] = \{1, \ldots, n\}$, edge ij independently present w.p. p_{ij}.
Example: Erdős-Rényi model, for which $p = \lambda/n$ for some $\lambda > 0$.

$\triangleright$ Configuration model: Vertices in $[n]$ have prescribed degree, graph constructed by pairing half-edges.

$\triangleright$ Preferential attachment model: Growing network, new vertices more likely to attach to old vertices having high degree.

Models typically are non-spatial and have small clustering.

AIM: construct simple spatial scale-free random graph model.

Inhomogeneous rgs

Norros-Reittu model: Equip each vertex $i \in [n] = \{1, \ldots, n\}$ with random weight W_i, where $(W_i)_{i \in [n]}$ are i.i.d. random variables.

Attach edge with probability p_{ij} between vertices i and j, where

$$p_{ij} = 1 - \mathrm{e}^{-\lambda W_i W_j / n}.$$

Different edges are conditionally independent given weights, and $\lambda > 0$ is parameter. Retrieve Erdős-Rényi RG with $p = 1 - \mathrm{e}^{-\lambda/n}$ when $W_i \equiv 1$.

$\triangleright$ Related models:
Chung-Lu model: $p_{ij} = (W_i W_j / n) \wedge 1$;
Generalized random graph: $p_{ij} = W_i W_j / (n + W_i W_j)$;

Janson (2010): Conditions for asymptotic equivalence.
Bollobás-Janson-Riordan (2007):
General set-up inhomogeneous random graphs.

Long-range percolation

Consider model on $\mathbb{Z}^d$ where we attach edge between $x, y \in \mathbb{Z}^d$ independently with probability

$$p_{x,y} = 1 - \mathrm{e}^{-\lambda/|x-y|^{\alpha}}.$$

Degree distribution:

$$D_x = \sum_{y \in \mathbb{Z}^d} I_{x,y},$$

with $I_{x,y}$ independent Bernoulli variables with success prob. p_{xy}.

Properties:

$\triangleright$ Percolation function continuous when $\alpha \in (d, 2d)$ (Berger 02);
$\triangleright$ Graph distances polylogarithmic when $\alpha \in (d, 2d)$ (Biskup 04);
$\triangleright$ Model has high clustering, i.e., many triangles;
$\triangleright$ Model never scale-free, i.e., either degrees are infinite a.s., or have thin tails;
$\triangleright$ Instantaneous percolation only when degrees are infinite a.s.

Percolation in random environment

$\triangleright$ Equip each vertex $x \in \mathbb{Z}^d$ with random weight W_x, where

$$(W_x)_{x \in \mathbb{Z}^d} \text{ are i.i.d. random variables.}$$

$\triangleright$ Conditionally on weights, edges in graph are independent, and probability that edge between x and y is present equals

$$p_{xy} = 1 - \mathrm{e}^{-\lambda W_x W_y / |x-y|^\alpha}.$$

$\triangleright$ Special attention to weights with power-law distribution:

$$\mathbb{P}(W_x \geq w) = w^{-(\tau-1)} L(w),$$

where $\tau > 1$, $w \mapsto L(w)$ is slowly varying. (Often take $L(w) \equiv c$.)

$\triangleright$ Long-range nature determined by parameter $\alpha > 0$.
$\triangleright$ Percolative properties determined by parameter $\lambda > 0$.
$\triangleright$ Inhomogeneity determined by distribution of (W_x).

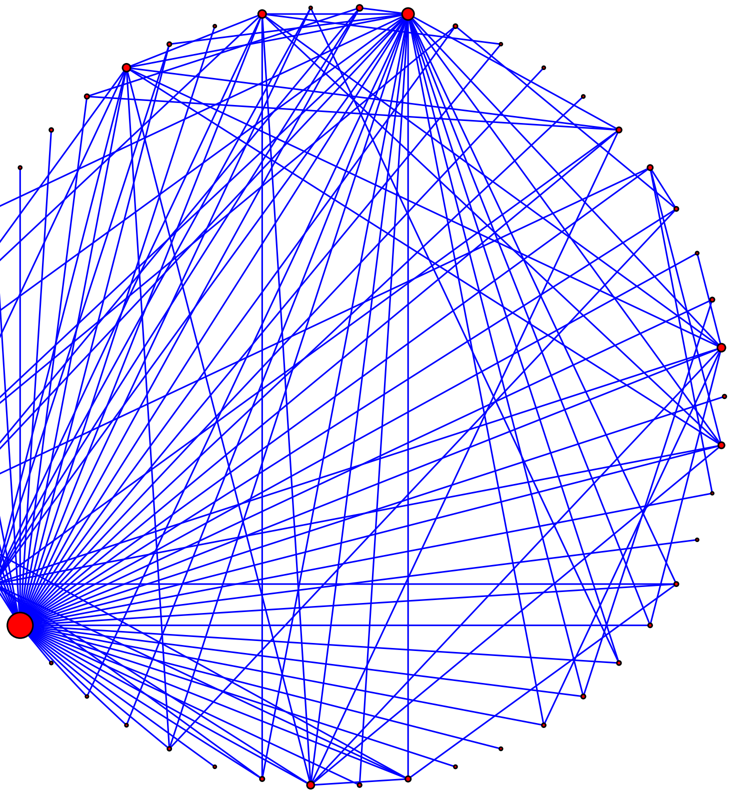
Questions and remarks

Model interpolates between

▷ long-range percolation, obtained when $W_x \equiv 1$;
▷ inhomogeneous random graphs, more precisely,
Poissonian random graph or Norros-Reittu model (06).
▷ small-world model (Strogatz-Watts) which has torus as vertex set, and rare macroscopic connections. We have connections on all length scales.

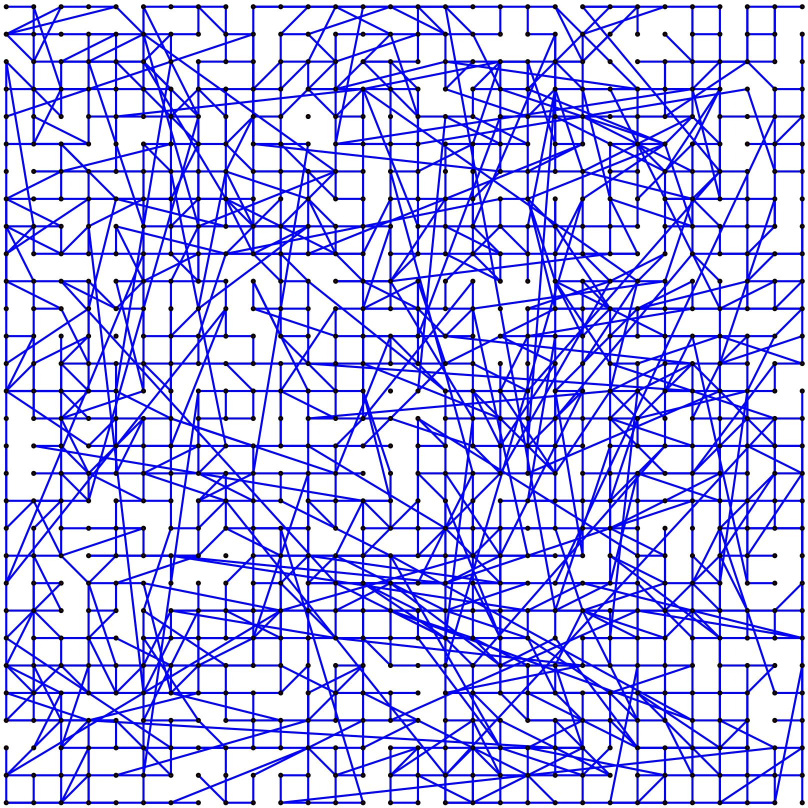
Investigate:
▷ Degree structure: How many neighbors do vertices have?
▷ Percolation: For which $\lambda > 0$ is there infinite component?
▷ Distances: What is graph distance x and y as $|x - y| \to \infty$?

Inhomogeneous RG



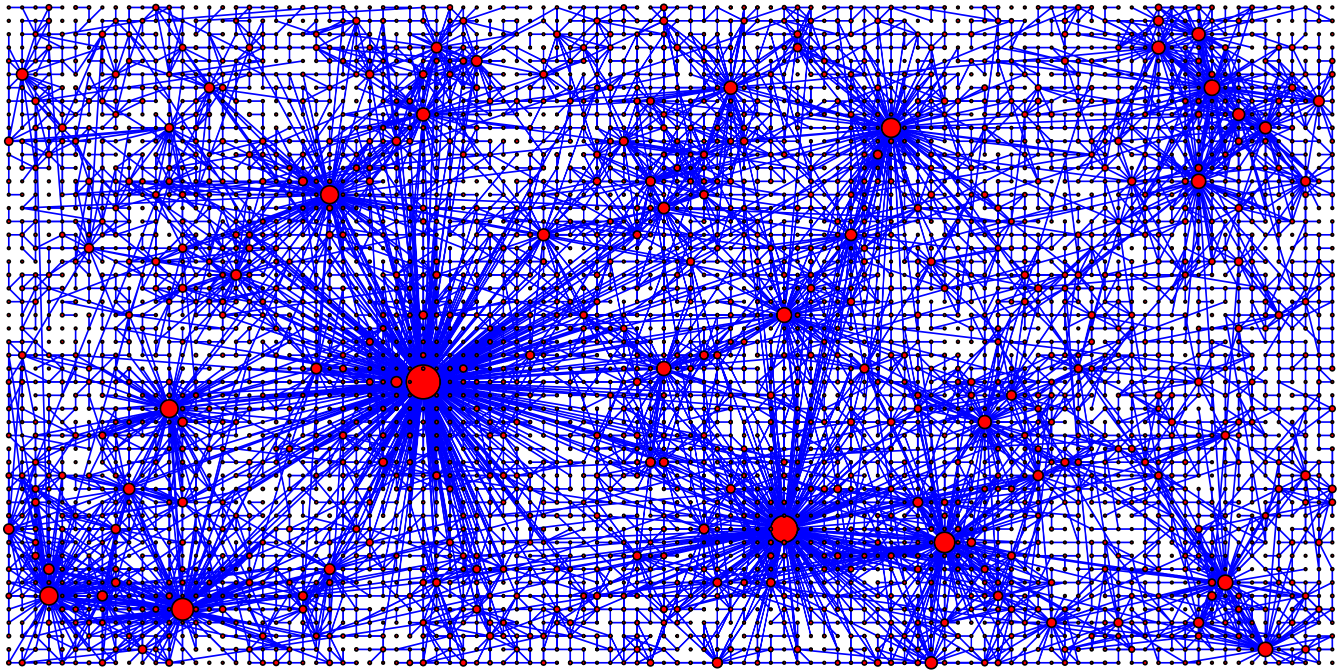
$\tau = 1.95$ (Joost Jorritsma)

Long-range percolation



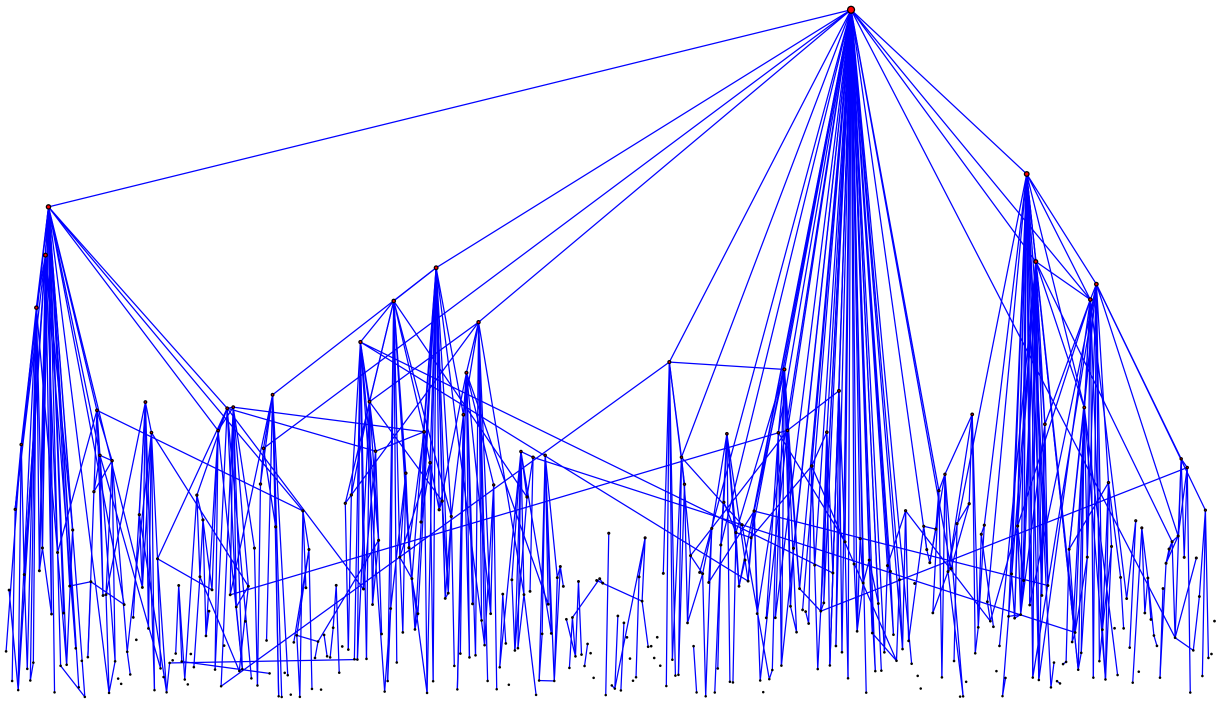
$d = 2,\ \alpha = 3.9,\ \lambda = 0.1$ (Joost Jorritsma)

Scale-free percolation



$d = 2,\ \alpha = 3.9,\ \tau = 1.95,\ \lambda = 0.1$ (Joost Jorritsma)

Scale-free percolation



$d = 1,\ \alpha = 2,\ \tau = 1.95,\ \lambda = 0.1$ (Joost Jorritsma)

Degrees

Special attention to weights with power-law distribution:

$$\mathbb{P}(W_x \geq w) = w^{-(\tau-1)} L(w),$$

where $\tau > 1$, $w \mapsto L(w)$ is slowly varying. (Often take $L(w) \equiv c$.)

Theorem 1 (Infinite degrees). $\mathbb{P}(D_0 = \infty \mid W_0 > 0) = 1$ when either $\alpha \leq d$, or $\alpha > d$ for power-law weights with $\gamma = \alpha(\tau - 1)/d < 1$.

Theorem 2 (Power-law degrees). For power-law weights, when $\alpha > d$ and $\gamma = \alpha(\tau - 1)/d > 1$, there exists a function $s \mapsto \ell(s)$ that is slowly varying at infinity s.t.

$$\mathbb{P}(D_0 > s) = s^{-\gamma} \ell(s).$$

Power-law degrees in percolation model:

Scale-free percolation.

Degrees: Proof Theorem 1

W.l.o.g. take $\lambda = 1$. First take $\alpha > d$, so that $\gamma = \alpha(\tau - 1)/d \leq 1$ implies $\tau \in (1, 2)$. For power-law weight distributions with $\tau \in (1, 2)$,

$$\mathbb{E}[W_y \mathbb{1}_{\{W_y \leq s\}}] = \Theta(s^{2-\tau}).$$

Thus, when $\gamma = \alpha(\tau - 1)/d \leq 1$, using $1 - \mathrm{e}^{-x} \geq x \mathbb{1}_{[0,1]}(x)/2$,

$$\begin{aligned}\sum_{y \neq 0} \mathbb{P}((0, y) \text{ occupied} \mid W_0 = w) &= \sum_{y \neq 0} \mathbb{E}\Big[1 - \mathrm{e}^{-wW_y/|y|^\alpha}\Big] \\ &\geq \frac{1}{2} \sum_{y \neq 0} \mathbb{E}\Big[wW_y/|y|^\alpha \mathbb{1}_{\{W_y \leq |y|^\alpha / w\}}\Big] \\ &\geq C w^{-(2-\tau)} \sum_{y \neq 0} \frac{1}{|y|^{\alpha(\tau-1)}} = \infty.\end{aligned}$$

By Borel-Cantelli, implies that $\mathbb{P}(D_0 = \infty | W_0 = w) = 1$ when $w > 0$. Similar (and easier) when $\alpha \leq d$.

Degrees: Proof Theorem 2

Crucially use that, for $\alpha > d$, as $a \to \infty$,

$$\sum_{y \neq 0} (1 - \mathrm{e}^{-a/|y|^\alpha}) = v_{d,\alpha} a^{d/\alpha} (1 + o(1)).$$

Thus, when $w > 1$ is large, and with $\xi = v_{d,\alpha} \mathbb{E}[W^{d/\alpha}] < \infty$,

$$\mathbb{E}[D_0 \mid W_0 = w] = \sum_{y \neq 0} \Big(1 - \mathbb{E}\Big[\mathrm{e}^{-wW_y/|y|^\alpha}\Big]\Big) \approx \xi w^{d/\alpha},$$

Conditionally on $W_0 = w$, D_0 is sum independent indicators, and thus highly concentrated when mean is large, i.e.,

$$\mathbb{P}(D_0 \geq s) \approx \mathbb{P}(W_0 \geq (s/\xi)^{\alpha/d}) \approx \ell(s) s^{-\alpha(\tau-1)/d} = \ell(s) s^{-\gamma}.$$

$\gamma > 1$: finite-mean degrees;
$\gamma > 2$: finite-variance degrees.

Percolation critical value

From now on, assume that long-range parameter $\alpha > d$ and power-law exponent $\gamma = \alpha(\tau - 1)/d > 1$.

Write $x \longleftrightarrow y$ when there is path of occupied bonds connecting x and y. Let $\mathcal{C}(x) = \{y \colon x \longleftrightarrow y\}$ be cluster of x.

$\triangleright$ Percolation probability: $\theta(\lambda) = \mathbb{P}(|\mathcal{C}(0)| = \infty)$.
$\triangleright$ Critical percolation value: $\lambda_c = \inf\{\lambda \colon \theta(\lambda) > 0\}$.

Theorem 3 (Finiteness critical value).
(a) $\lambda_c < \infty$ in $d \geq 2$ if $\mathbb{P}(W = 0) < 1$.
(b) $\lambda_c < \infty$ in $d = 1$ if $\alpha \in (1, 2]$, $\mathbb{P}(W = 0) < 1$.
(c) $\lambda_c = \infty$ in $d = 1$ if $\alpha > 2, \gamma = \alpha(\tau - 1)/d > 2$.

Positivity threshold

Theorem 4 (Positivity critical value).

$\lambda_c > 0$ when $\gamma = \alpha(\tau - 1)/d > 2.$

Theorem 5 (Zero critical value).

$\lambda_c = 0$ when $\gamma = \alpha(\tau - 1)/d \in (1, 2)$, i.e., $\theta(\lambda) > 0$ for every $\lambda > 0.$

Robustness of phase transition (Jacob, Mörters)

Identical to Norros-Reittu model, novel for percolation models:

Norros-Reittu model: $G = K_n, p_{ij} = 1 - \mathrm{e}^{-\lambda W_i W_j / n}$. Giant component exists for every $\lambda > 0$ when variance degrees is infinite.
NR-model: degrees have same number of moments as weights W.

Proof Theorem 4

We first assume that for $\mathbb{E}[W^2] < \infty$. When $|\mathcal{C}(0)| = \infty$, there exists paths of arbitrary length from origin:

$$\theta(\lambda) \leq \sum_{x_1,\dots,x_n} \mathbb{P}((x_{i-1}, x_i) \text{ occupied}) = \sum_{x_1,\dots,x_n} \mathbb{E}\Big[\prod_{i=1}^{n} p_{x_{i-1},x_i}\Big],$$

where sum is over distinct vertices, with $x_0 = 0$. Bound

$$p_{x,y} = 1 - \mathrm{e}^{-\lambda W_x W_y |x-y|^{-\alpha}} \leq \lambda W_x W_y |x-y|^{-\alpha} :$$

$$\begin{aligned}\theta(\lambda) &\leq \lambda^n \sum_{x_1,\dots,x_n} \mathbb{E}\Big[\prod_{i=1}^{n} W_{x_{i-1}} W_{x_i} |x_{i-1} - x_i|^{-\alpha}\Big] \\ &= \lambda^n \sum_{x_1,\dots,x_n} \mathbb{E}[W]^2 \mathbb{E}[W^2]^{n-1} \prod_{i=1}^{n} |x_{i-1} - x_i|^{-\alpha} \leq \Big(\lambda \mathbb{E}[W^2] \sum_{x \neq 0} |x|^{-\alpha}\Big)^n.\end{aligned}$$

Proof Theorem 4

When $\mathbb{E}[W^2] = \infty$, instead use Cauchy-Schwarz and bound

$$p_{x,y} = 1 - \mathrm{e}^{-\lambda W_x W_y |x-y|^{-\alpha}} \leq \Big(\lambda W_x W_y |x-y|^{-\alpha} \wedge 1\Big) :$$

$$\theta(\lambda) \leq \sum_{x_1,\ldots,x_n} \mathbb{E}\Big[\prod_{i=1}^{n} \Big(\lambda W_{x_{i-1}} W_{x_i} |x_{i-1} - x_i|^{-\alpha} \wedge 1\Big)\Big]$$

$$\leq \Big(\sum_{x\neq 0} \mathbb{E}\Big[\Big(\lambda W_0 W_1 |x|^{-\alpha} \wedge 1\Big)^2\Big]^{1/2}\Big)^n.$$

Key estimate: if $\mathbb{P}(W \geq w) \leq c w^{-(\tau-1)}$ with $\tau \in (1,3)$, then

$$g(u) \equiv \mathbb{E}\Big[\Big(W_1 W_2/u \wedge 1\Big)^2\Big] \leq C(1 + \log u) u^{-(\tau-1)}.$$

$\alpha(\tau-1)/2 > d$ when $\gamma = \alpha(\tau-1)/d > 2$, so above sum finite.

Proof Theorem 5

We use renormalization argument for $\gamma \in (1,2)$. Prove $\theta(\lambda) > 0$ for any $\lambda > 0$ small. Take r_λ large. By extreme value theory,

$$\max_{|x|<r_\lambda} W_x = \Theta_{\mathbb{P}}(r_\lambda^{d/(\tau-1)}).$$

For $x \in \mathbb{Z}^d$, let $x(\lambda)$ be maximal weight vertex in $\{y\colon |y - r_\lambda x| \leq r_\lambda\}$. Say (x,y) occupied when $(x(\lambda), y(\lambda))$ occupied.

For nearest-neighbor x, y, and with high probability,

$$\mathbb{P}((x,y) \text{ occ.} \mid (W_x)_{x\in\mathbb{Z}^d}) \approx 1 - \mathrm{e}^{-\lambda W_{x(\lambda)} W_{y(\lambda)} r_\lambda^{-\alpha}} \approx 1 - \mathrm{e}^{-\lambda r_\lambda^{2d/(\tau-1)-\alpha}}.$$

Note $2d/(\tau-1) - \alpha > 0$ precisely when $\gamma = \alpha(\tau-1)/d < 2$.

Take r_λ so large that $\lambda r_\lambda^{2d/(\tau-1)-\alpha} \gg 1$. Then nearest-neighbor percolation model supercritical for small $\lambda > 0$. Implies that $\theta(\lambda) > 0$.

Distances

Theorem 6 (Loglog distances for infinite variance degrees).
Fix $\lambda > 0$. For $\gamma \in (1,2)$ and any $\eta > 0$,

$$\lim_{|x|\to\infty} \mathbb{P}\Big(d(0,x) \leq (1+\eta)\frac{2\log\log|x|}{|\log(\gamma-1)|}\Big| 0 \longleftrightarrow x\Big) = 1.$$

and

$$\lim_{|x|\to\infty} \mathbb{P}\Big(d(0,x) \geq (1-\eta)\frac{2\log\log|x|}{|\log(\kappa)|}\Big) = 1,$$

where $\kappa = (\gamma \wedge \alpha/d) - 1.$

Identical to distance results for Norros-Reittu model (Chung-Lu 06, Norros-Reittu 06).

Distances

Theorem 7. (Logarithmic bounds for finite variance degrees)
Fix $\lambda > \lambda_c$. For $\gamma = \alpha(\tau - 1)/d > 2$, there exists an $\eta > 0$ such that

$$\lim_{|x|\to\infty} \mathbb{P}(d(0,x) \geq \eta \log |x|) = 1.$$

$\triangleright$ Phase transition for distances depending on whether degrees have finite or infinite variance.

Theorem 8 (Polynomial lower bound distances).
Fix $\lambda > \lambda_c$. For $\gamma = \alpha(\tau - 1)/d > 2$ and $\alpha > 2d$, there exists $\varepsilon > 0$ such that

$$\lim_{|x|\to\infty} \mathbb{P}(d(0,x) \geq |x|^{\varepsilon}) = 1.$$

$\triangleright$ Similar to long-range percolation (Biskup 04, Berger 04).

Further results

$\triangleright$ Diameter for $\alpha < d$ or $\gamma < 1.$
Benjamini, Kesten, Peres and Schramm (04): For long-random percolation, $\mathrm{diam}(\mathcal{C}_\infty) = \lceil d/(d-\alpha) \rceil$ a.s.
Heydenreich, Hulshof, Jorritsma (16): diameter bounded.

$\triangleright$ Random walk on scale-free percolation cluster:
Heydenreich, Hulshof, Jorritsma (16):
Transient when $\alpha \in (d, 2d)$ or $\gamma \in (1, 2).$
Recurrent when $d = 2$ and $\gamma > 2$ or $\tau > 2.$

Open problems

▷ Critical behavior:
Continuity percolation function?
Hazra+Wütrich (14): Yes, for $\alpha \in (d, 2d)$.
What is upper-critical dimension?

Norros-Reittu model: Scaling limit same as for Erdős-Rényi random graph when $\gamma > 3$, different when $\gamma \in (2, 3)$. (BvdHvL(09a,b)).

▷ Distances:
What happens when $\alpha > 2d, \gamma > 2$?
Precise behavior for $\alpha \in (d, 2d), \gamma > 2$? Polylogarithmic as for long-range percolation: Biskup (04): $(\log |x|)^{\Delta}$, where $\Delta = \log_2(2d/\alpha)$?

Hazra+Wütrich (14): Bounded below and above by $(\log |x|)^{\Delta}$ for different Δ.

Open problems

$\triangleright$ Other spatial models: Deprez+Hazra+Wüthrich (15), Hirsch (14): Poisson version on $\mathbb{R}^d$. Results on torus?

Can one define a spatial preferential attachment model on $\mathbb{Z}^d$?
On torus: Work by Jordan (10), Flaxman, Frieze, Vera (06,07): Focus is on degree sequence.
SPAM: Janssen, Pralat, Wilson (11): also geometry investigated.
Jacob, Mörters (15): Robustness!

Spatial configuration model on $\mathbb{Z}^d$?
Deijfen and collaborators: matching problems and percolation.

Literature

[1] Berger. Transience, recurrence and critical behavior for long-range percolation. Preprint, arXiv:math/0409021v1, (2004).

[2] Berger. A lower bound for chemical distances in sparse long-range percolation models. *Comm. Math. Phys.* **226**, 531–558, (2002).

[3] Biskup. On the scaling of the chemical distance in long-range percolation models, *Ann. Probab.* **32**, 2933–2977, (2004).

[4] Deijfen, van der Hofstad and Hooghiemstra.
Scale-free percolation. *AIHP* **49**(3), 817–838, (2013).

[5] Norros and Reittu. On a conditionally Poissonian graph process. *Adv. Appl. Probab.* **38**, 59–75, (2006).

[6] Deprez, Hazra and Wütrich. Inhomogeneous long-range percolation for real-life network modeling. Risks **3**: 1-23, (2015).

[7] Hazra and Wütrich. Continuity of the percolation probability & chemical distances in inhomogeneous long-range percolation. Preprint (2014).

[8] Hirsch. From heavy-tailed Boolean models to scale-free Gilbert graphs. Preprint (2014).

And now for something completely different...

Network Pages: interactive website by and for network afficionados...

`www.networkpages.nl`

We welcome contributions from everyone!